Recovery of the Dirichlet-to-Neumann map from scattering data in the plane (Harmonic Analysis and Nonlinear Partial Differential Equations)

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By

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Abstract

The differences between plane waves with fixed frequency and their analogues that have been distorted by a potential V yield the *scattering amplitude.* It is a classical problem to recover V from this information. It is well-known that the scattering amplitude uniquely determines the *Dirichlet-to-Neumann (DN) map* (from which the potential can be recovered) and there are a number of different approaches to proving this. Here we provide explicit formulae, closely following the work of Nachman and Stefanov, which recover the DN map from the scattering amplitude in the plane.

§ 1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain which contains the support of a real potential V. We suppose throughout that $V \in L^p(\Omega)$, with $p > 2$, and that $k^2 > 0$ is not a Dirichlet eigenvalue for the Hamiltonian $-\Delta + V$. Then the Dirichlet-to-Neumann (DN) map Λ_V can be formally defined by

$$\Lambda_V \; : \; u|_{\partial\Omega} \mapsto \nabla u \cdot n|_{\partial\Omega},$$

where u is the solution to the Schrödinger equation

$$(-\Delta + V)u = k^2 u \tag{1.1}$$

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with the given Dirichlet data. In recent works (see for example [14, 11, 4, 2]), methods for recovering potentials from their DN maps have been developed.

On the other hand, for each $\theta \in \mathbb{S}^1$, the outgoing scattering solutions of (1.1) solve the Lippmann–Schwinger equation;

$$u(x,\theta) = e^{ikx\cdot\theta} - \int_{\mathbb{R}^2} G_0(x,y)V(y)u(y,\theta)\,dy.$$

Here, G_0 is the outgoing Green's function which satisfies

$$(-\Delta - k^2)G_0(x,y) = \delta(x-y). \tag{1.2}$$

It can be calculated explicitly and is nothing more than a constant multiple of the zeroth order Hankel function of the first kind.

The scattering solutions satisfy the 'asymptotics' $u(x,\theta) = e^{ikx\cdot\theta}$ if $V = 0$. That is they measure how much the plane wave has been distorted by the potential. Indeed, later we will derive the asymptotics

$$G_0(x,y) = e^{-ik\frac{x}{|x|}\cdot y}\frac{e^{i\frac{\pi}{4}}e^{ik|x|}}{\sqrt{8\pi k|x|}} + o\Big(\frac{1}{\sqrt{|x|}}\Big),$$

so that, by plugging into the Lippmann–Schwinger equation, we obtain

$$u(x,\theta) = e^{ik\theta\cdot x} - A_V\Big(\frac{x}{|x|},\theta\Big)\frac{e^{i\frac{\pi}{4}}e^{ik|x|}}{\sqrt{8\pi k|x|}} + o\Big(\frac{1}{\sqrt{|x|}}\Big),$$

Here, the scattering amplitude $A_V : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{C}$ satisfies

$$A_V(\vartheta,\theta) = \int_{\mathbb{R}^2} e^{-ik\vartheta\cdot y}V(y)u(y,\theta)\,dy, \tag{1.3}$$

and a classical problem is to recover the potential from this information alone.

It is well-known that the scattering amplitude uniquely determines the DN map (and *vice versa*) and so solutions to the first question, regarding the DN map, also provide solutions to the scattering question. There are a number of different approaches to showing this equivalence (see for example [3, 9, 16, 14, 15]). Here we provided explicit formulae, initially following an argument due to Nachman [10, Section 3] and then adapting three–dimensional arguments, due to Stefanov [13], to the two–dimensional problem.

§ 2. The formulae

It is well–known that, under the hypotheses of the introduction, for each $f \in H^{1/2}(\partial\Omega)$, there is a unique weak solution to the Dirichlet problem

$$\begin{cases} \Delta u = (V-k^2)u \\ u\big|_{\partial\Omega} = f, \end{cases} \tag{2.1}$$

that satisfies

$$\|u\|_{H^1(\Omega)} \leq C\|f\|_{H^{1/2}(\partial\Omega)} \tag{2.2}$$

(see for example [6] - in two dimensions $L^{n/2}(\mathbb{R}^n)$ can be replaced by $L^p(\mathbb{R}^2)$ with $p > 1$). Here $H^{1/2}(\partial\Omega) := H^1(\Omega)/H_0^1(\Omega)$, where $H_0^1(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. The DN map Λ_V is then defined by

$$\Big\langle \Lambda_V[f], \psi \Big\rangle = \int_{\partial\Omega} \Lambda_V[f]\,\psi = \int_\Omega V u\Psi + \nabla u \cdot \nabla\Psi$$

for any $\Psi \in H^1(\Omega)$ with $\psi = \Psi + H_0^1(\Omega)$. When the solution and boundary are sufficiently regular, this definition coincides with that of the introduction by Green's formula. To see that Λ_V maps from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$, the dual of $H^{1/2}(\partial\Omega)$, we note that by Hölder's inequality and the Hardy–Littlewood–Sobolev inequality,

$$\begin{aligned}\Big|\Big\langle \Lambda_V[f], \psi \Big\rangle\Big| &\leq \|u\|_{H^1(\Omega)}\|\Psi\|_{H^1(\Omega)} + \|V\|_p\|u\|_{L^q(\Omega)}\|\Psi\|_{L^q(\Omega)} \\ &\leq (1 + C\|V\|_p)\|u\|_{H^1(\Omega)}\|\Psi\|_{H^1(\Omega)}\end{aligned}$$

whenever $\Psi \in H^1(\Omega)$. Here $\frac{1}{p} + \frac{2}{q} = 1$ with $p > 1$. By (2.2), we obtain

$$\Big|\Big\langle \Lambda_V[f], \psi \Big\rangle\Big| \leq C(1 + \|V\|_p)\|f\|_{H^{1/2}(\partial\Omega)}\|\psi\|_{H^{1/2}(\partial\Omega)}$$

and so the DN map is bounded.

Essential to our analysis will be the outgoing Green's function G_V that satisfies

$$(-\Delta + V - k^2)G_V(x, y) = \delta(x - y) \tag{2.3}$$

and the corresponding near-field operator S_V defined via the single layer potential

$$S_V[f](x) = \int_{\partial\Omega} G_V(x, y) f(y)\, dy.$$

This is a bounded and invertible mapping from $H^{-1/2}(\partial\Omega)$ to $H^{1/2}(\partial\Omega)$ (the two–dimensional proof can be found in [7, Proposition A.1]). Then Nachman's formula [9],

$$\Lambda_V = \Lambda_0 + S_V^{-1} - S_0^{-1},$$

allows us to recover the DN map as soon as we recover the single layer potential S_V from the scattering amplitude A_V at energy k^2.

When Ω is a disc, Nachman recovered S_V via formulae given by expansions in spherical harmonics as below. Otherwise he used a density argument (we remark that Sylvester [15] also invokes density in order to recover). Since it is occasionally convenient to work with different domains (see for example [2] where it is convenient to work on a square), at this point we follow instead an argument of Stefanov [13], obtaining an explicit formula for the Green's function G_V in terms of A_V. Alternatively it seems likely that one could pass to the DN map on other domains from the the DN map on the disc via the argument in [11, Section 6] for the conductivity problem, however we prefer this more direct approach. We recover G_V outside of a disc which contains the potential, but which is properly contained in Ω, so that S_V can be obtained by integrating along the boundary $\partial\Omega$.

First we require the following well-known asymptotics.

Lemma 2.1. *Let $V \in L^p(\Omega)$ with $p > 2$. Then*

$$G_V(x,y) - G_0(x,y) = \frac{-i}{8\pi k}\frac{e^{ik|x|}}{|x|^{\frac{1}{2}}}\frac{e^{ik|y|}}{|y|^{\frac{1}{2}}}A_V\Big(-\frac{x}{|x|},\frac{y}{|y|}\Big) + o\Big(\frac{1}{|x|^{\frac{1}{2}}|y|^{\frac{1}{2}}}\Big).$$

Proof. Using the asymptotics of the Hankel function for large r;

$$H_0^{(1)}(r) = e^{-i\frac{\pi}{4}}\Big(\frac{2}{\pi r}\Big)^{\frac{1}{2}}e^{ir} + o\Big(\frac{1}{r^{\frac{1}{2}}}\Big)$$

(see for example [8, Section 5.16] or [5, pp. 66]) and the Taylor expansion at a fixed $y \in \mathbb{R}^2$,

$$|x-y| = |x|\Big(1 - 2\tfrac{x}{|x|^2}\cdot y + \tfrac{|y|^2}{|x|^2}\Big)^{1/2} = |x| - \tfrac{x}{|x|}\cdot y + O\Big(\frac{1}{|x|}\Big), \qquad |x| \gg |y|,$$

one obtains the asymptotic formula

$$G_0(x,z) = \frac{i}{4}H_0^{(1)}(k|x-y|) = \frac{e^{i\frac{\pi}{4}}}{(8\pi)^{\frac{1}{2}}}\frac{e^{ik|x|}}{k^{\frac{1}{2}}|x|^{\frac{1}{2}}}e^{-ik\frac{x}{|x|}\cdot z} + o\Big(\frac{1}{|x|^{\frac{1}{2}}}\Big). \tag{2.4}$$

On the other hand, given that the outgoing solution to (2.3) is unique, one can verify that

$$G_V(x,y) = G_0(x,y) - \int_{\mathbb{R}^2} G_0(x,z)V(z)G_V(y,z)\,dz. \tag{2.5}$$

For this one must check that the outgoing Sommerfeld radiation condition is also uniformly satisfied for y in compact sets by the right-hand side. This follows from the fact that G_0 satisfies the condition (see for example [12, Proposition 2.1]) and that the resolvent

$$(-\Delta + V - k^2 - i0)^{-1} : V \mapsto \int_{\mathbb{R}^2} V(z)G_V(\cdot,z)\,dz$$

is bounded from $L^2((1+|\cdot|^2)^{\delta})$ to $L^2((1+|\cdot|^2)^{-\delta})$ with $\delta > 1/2$ (see [1, Theorem 4.2]).

Similarly, the outgoing scattering solutions of (1.1), in the direction $-\frac{x}{|x|}$, solve

$$u(y, -\tfrac{x}{|x|}) = e^{-ik\frac{x}{|x|}\cdot y} - \int_{\mathbb{R}^2} e^{-ik\frac{x}{|x|}\cdot z}V(z)G_V(y,z)\,dy. \tag{2.6}$$

Plugging (2.4) into (2.5) and comparing with (2.6), we see that G_V satisfies

$$G_V(x,z) = \frac{e^{i\frac{\pi}{4}}}{(8\pi)^{\frac{1}{2}}} \frac{e^{ik|x|}}{k^{\frac{1}{2}}|x|^{\frac{1}{2}}} u\Big(z, -\frac{x}{|x|}\Big) + o\Big(\frac{1}{|x|^{\frac{1}{2}}}\Big). \tag{2.7}$$

As in (2.5), we also have that

$$G_V(x,y) - G_0(x,y) = -\int_{\mathbb{R}^2} G_0(y,z)V(z)G_V(x,z)\,dz. \tag{2.8}$$

Substituting (2.4) and (2.7) into this, we see that $G_V(x,y) - G_0(x,y)$ is equal to

$$\frac{-i}{8\pi k}\frac{e^{ik|x|}}{|x|^{\frac{1}{2}}}\frac{e^{ik|y|}}{|y|^{\frac{1}{2}}}\int_{\mathbb{R}^2} e^{-ik\frac{y}{|y|}\cdot z}V(z)u\Big(z,-\frac{x}{|x|}\Big)dz + o\Big(\frac{1}{|x|^{\frac{1}{2}}|y|^{\frac{1}{2}}}\Big),$$

so that, by using the formula (1.3), we obtain the result. □

In the following theorem, $H_n^{(1)}$ denotes the Hankel function of the first kind and nth order (see for example [8] or [5]) and we write x in polar coordinates as $(|x|, \phi_x)$.

Theorem 2.2. *Let $V \in L^p(\Omega)$, with $p > 2$, be supported in the disc of radius ρ, centred at the origin, and consider its Fourier series*

$$A_V(\vartheta,\theta) = \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} a_{n,m}e^{in\phi_\vartheta}e^{im\phi_\theta}.$$

Then

$$G_V(x,y) - G_0(x,y) = \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \frac{(-1)^n}{16} i^{n+m} a_{n,m} H_n^{(1)}(k|x|)H_m^{(1)}(k|y|)e^{in\phi_x}e^{im\phi_y},$$

where the series is uniformly and absolutely convergent for $|x| > |y| > R > \frac{3}{2}\rho > 0$.

Proof. We can expand $H_0^{(1)}(k|x-y|)$ as

$$H_0^{(1)}(k|x-y|) = H_0^{(1)}(k|x|)J_0(k|y|) + 2\sum_{n\geq 1} H_n^{(1)}(k|x|)J_n(k|y|)\cos(\phi_x - \phi_y),$$

(see for example [5, Section 3.4] or [12, Theorem 3.4]). As $H_{-n}^{(1)} = (-1)^n H_n^{(1)}$ and $J_{-n} = (-1)^n J_n$, in order to separate variables it will be convenient to write this as

$$G_0(x,y) = \frac{i}{4}\sum_{n\in\mathbb{Z}} H_n^{(1)}(k|x|)J_n(k|y|)e^{in\phi_x}e^{-in\phi_y}.$$

Substituting (2.5) into (2.8) we obtain $G_V - G_0 = -I_1 + I_2$, where

$$I_1 = \int_{\mathbb{R}^2} G_0(x,z)V(z)G_0(z,y)\,dz$$

$$I_2 = \int_{\mathbb{R}^2} G_0(x,z_1)V(z_1)\int_{\mathbb{R}^2} G_V(z_1,z_2)V(z_2)G_0(y,z_2)\,dz_1dz_2.$$

Now in both integrals we introduce the expansion of G_0 (recall that $G_0(z,y) = G_0(y,z)$), extracting the terms independent of z, z_1, z_2. In this way we get

$$I_1 = \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \alpha_{n,m} H_n^{(1)}(k|x|)H_m^{(1)}(k|y|)e^{in\phi_x}e^{im\phi_y}, \tag{2.9}$$

$$I_2 = \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \beta_{n,m} H_n^{(1)}(k|x|)H_m^{(1)}(k|y|)e^{in\phi_x}e^{im\phi_y}, \tag{2.10}$$

where

$$\begin{aligned}\alpha_{n,m} &= -\frac{1}{16}\int_{\mathbb{R}^2} V(z)J_n(k|z|)J_m(k|z|)e^{-i(n+m)\phi_z}\,dz,\\ \beta_{n,m} &= -\frac{1}{16}\int_{\mathbb{R}^4} J_n(k|z_1|)V(z_1)G_V(z_1,z_2)V(z_2)J_m(k|z_2|)e^{-in\phi_{z_1}}e^{-im\phi_{z_2}}\,dz_1dz_2.\end{aligned}$$

It remains to show that the sums (2.9) and (2.10) converge uniformly and absolutely for $|x| > |y| > R > \frac{3}{2}\rho$. Once we know that this is the case, we can take limits and use the asymptotics of the Hankel functions for large r;

$$H_n^{(1)}(r) = e^{-i(n\frac{\pi}{2}+\frac{\pi}{4})}\Big(\frac{2}{\pi r}\Big)^{\frac{1}{2}}e^{ir} + o\Big(\frac{1}{r^{\frac{1}{2}}}\Big)$$

(see for example [8, Section 5.16] or [5, pp. 66]), and then Lemma 2.1 tells us that

$$(-i)^{n+m+1}\frac{2}{\pi k}(\beta_{n,m} - \alpha_{n,m}) = -i\frac{(-1)^n}{8\pi k}a_{n,m}.$$

To see that the sums converge note that, by Hölder's inequality, we have

$$\begin{aligned}|\alpha_{n,m}| &\le C_\rho \|V\|_{L^p}\|J_n(k|\cdot|)\|_{L^\infty(B_\rho)}\|J_m(k|\cdot|)\|_{L^\infty(B_\rho)},\\ |\beta_{n,m}| &\le \|G_V\|_{L^2(B_\rho\times B_\rho)}\|V\|_{L^p}^2\|J_n(k|\cdot|)\|_{L^\infty(B_\rho)}\|J_m(k|\cdot|)\|_{L^\infty(B_\rho)}.\end{aligned}$$

At this point we deviate from [13] as there seems to be less local knowledge regarding G_V in two dimensions. Instead we can rewrite (2.8) as

$$G_V(\cdot,y) = G_0(\cdot,y) - (-\Delta + V - k^2 - i0)^{-1}[VG_0(\cdot,y)],$$

and use that the resolvent is bounded from $L^2((1+|\cdot|^2)^\delta)$ to $L^2((1+|\cdot|^2)^{-\delta})$ with $\delta > 1/2$ (see [1, Theorem 4.2]). Thus, using that V is compactly supported, and taking $\frac{1}{2} = \frac{1}{p} + \frac{1}{q}$ with sufficiently large q,

$$\begin{aligned}\|G_V(\cdot,y)\|_{L^2(B_\rho)} &\le \|G_0(\cdot,y)\|_{L^2(B_\rho)} + C_\rho\|VG_0(\cdot,y)\|_{L^2(B_\rho)}\\ &\le \|G_0(\cdot,y)\|_{L^2(B_\rho)} + C_\rho\|V\|_p\|G_0(\cdot,y)\|_{L^q(B_\rho)}.\end{aligned}$$

Integrating again with respect to y, and recalling that the singularity of $H_0^{(1)}$ at the origin is logarithmic, we see that $\|G_V\|_{L^2(B_\rho\times B_\rho)} \le C_\rho(1+\|V\|_p)$. Then, using the Taylor series expansion for the Bessel function,

$$|J_n(r)| = \Big|\sum_{j\ge 0}\frac{(-1)^j}{j!(|n|+j)!}\Big(\frac{r}{2}\Big)^{2j+|n|}\Big| \le \frac{1}{|n|!}\Big(\frac{\rho}{2}\Big)^{|n|}$$

with $0\le r\le\rho$, we see that

$$\begin{aligned}|\alpha_{n,m}| &\le C_\rho\|V\|_p\frac{1}{|n|!}\Big(\frac{k\rho}{2}\Big)^{|n|}\frac{1}{|m|!}\Big(\frac{k\rho}{2}\Big)^{|m|},\\ |\beta_{n,m}| &\le C_\rho(1+\|V\|_p)_p^3\frac{1}{|n|!}\Big(\frac{k\rho}{2}\Big)^{|n|}\frac{1}{|m|!}\Big(\frac{k\rho}{2}\Big)^{|m|}.\end{aligned}$$

Plugging these estimates, along with the forthcoming estimate (2.11) for the Hankel functions, into the sums (2.9) and (2.10), we see that they are bounded by constant multiples of

$$\sum_{n\ge 0}\sum_{m\ge 0}\Big(\frac{3\rho}{2R}\Big)^n\Big(\frac{3\rho}{2R}\Big)^m$$

provided that $|x|>|y|>R>0$. This series is of course convergent when $R>\frac{3}{2}\rho$, and so we are done. $\square$

For a fixed order n, the Hankel function of the first kind is well-known to decay at infinity as we saw earlier, however we need a bound that is uniform in n. The decay is not uniform in n as the singularity at the origin widens as n grows, however the following estimate is sufficient for our purposes.

Lemma 2.3. *For all $n\in\mathbb{Z}$ and all $r\ge R>0$,*

$$|H_n^{(1)}(r)| \le C_R|n|!\Big(\frac{3}{R}\Big)^{|n|}. \tag{2.11}$$

Proof. The Hankel functions have only one singularity, at the origin, and they decay at infinity (see for example [8, Section 5] or [5, Section 3.4]), so by taking the constant C_R sufficiently large, it will suffice to prove the estimate for $|n|\ge 2(R+6)$. As $H_{-n}^{(1)}=(-1)^nH_n^{(1)}$, we need only consider positive n and we divide these into two cases.

First we consider the easier case $4n(n-1)\le r^2$ and use the recurrence relation

$$H_{n-1}^{(1)}(r)+H_{n+1}^{(1)}(r)=\frac{2n}{r}H_n^{(1)}(r) \tag{2.12}$$

(see for example [8, Section 5.4]) to conclude that

$$\begin{aligned}|H_{n+1}^{(1)}(r)| &= \Big|\frac{2n}{r}\Big(\frac{2(n-1)}{r}H_{n-1}^{(1)}(r)-H_{n-2}^{(1)}(r)\Big)\Big) - H_{n-1}^{(1)}(r)\Big|\\ &\le |H_{n-1}^{(1)}(r)|+|H_{n-2}^{(1)}(r)|.\end{aligned}$$

Iterating this step, we see that

$$|H_n^{(1)}(r)| \leq 2^n\Big(|H_0^{(1)}(r)| + |H_1^{(1)}(r)| + |H_2^{(1)}(r)|\Big), \qquad 4n(n-1) \leq r^2.$$

Taking $C_R = \sup_{r\geq R}\big(|H_0^{(1)}(r)| + |H_1^{(1)}(r)| + |H_2^{(1)}(r)|\big)$, and recalling that we can suppose that $n! \geq (2R/3)^n$ (by Stirling's formula $n > 2R$ is a stronger assumption), we obtain

$$|H_n^{(1)}(r)| \leq C_R n!\Big(\frac{3}{R}\Big)^n, \qquad 4n(n-1) \leq r^2. \tag{2.13}$$

Next we consider the harder case $4n(n-1) > r^2$ and again use the recurrence relation (2.12) to conclude that

$$\begin{aligned}\frac{2n}{r}|H_n^{(1)}(r)| + |H_{n-1}^{(1)}(r)| &= \frac{2n}{r}\Big|\frac{2(n-1)}{r}H_{n-1}^{(1)}(r) - H_{n-2}^{(1)}(r)\Big| + |H_{n-1}^{(1)}(r)|\\ &\leq \frac{3n}{r}\Big(\frac{2(n-1)}{r}|H_{n-1}^{(1)}(r)| + |H_{n-2}^{(1)}(r)|\Big).\end{aligned}$$

Letting n_0 denote the smallest integer for which $4n_0(n_0-1) > r^2$, we iterate this inequality;

$$|H_{n+1}^{(1)}(r)| \leq \frac{3n}{r}\dots\frac{3n_0}{r}\Big(\frac{2(n_0-1)}{r}|H_{n_0-1}^{(1)}(r)| + |H_{n_0-2}^{(1)}(r)|\Big). \tag{2.14}$$

Now as long as $n_0 \geq 12$ (if $n_0 < 12$ then we are already done), we have

$$\frac{r}{2n_0} < \frac{2(n_0-1)}{r} \leq \frac{r}{2(n_0-2)} \leq \frac{6}{5}\frac{r}{2n_0},$$

which implies that $1/3 < 2(n_0-1)/r < 3$. For convenience we divide the first term and multiply the second term by this factor, so that

$$|H_{n+1}^{(1)}(r)| \leq 3\frac{3n}{r}\dots\frac{3n_0}{r}\Big(|H_{n_0-1}^{(1)}(r)| + \frac{2(n_0-1)}{r}|H_{n_0-2}^{(1)}(r)|\Big).$$

Now as

$$\frac{3n}{r}\dots\frac{3n_0}{r} \leq \frac{3n}{R}\dots\frac{3n_0}{R} = n!\Big(\frac{3}{R}\Big)^n\frac{1}{(n_0-1)!}\Big(\frac{R}{3}\Big)^{n_0-1}$$

this yields

$$|H_{n+1}^{(1)}(r)| \leq 3n!\Big(\frac{3}{R}\Big)^n\Big(\frac{1}{(n_0-1)!}\Big(\frac{R}{3}\Big)^{n_0-1}|H_{n_0-1}^{(1)}(r)| + \frac{1}{(n_0-2)!}\Big(\frac{R}{3}\Big)^{n_0-2}|H_{n_0-2}^{(1)}(r)|\Big).$$

As n_0 was chosen to be the smallest integer belonging to the second case, we can bound $|H_{n_0-1}^{(1)}(r)|$ and $|H_{n_0-2}^{(1)}(r)|$ using the estimate (2.13) from the first case. This yields the estimate for the second case, and so the proof is complete. $\square$

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